M2201 Exam. 2011 Solutions.

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Section A.

1. (a) [5 MARKS] Euclidean algorithm:

$$96 = 2 \cdot 40 + 16$$
$$40 = 2 \cdot 16 + 8$$
$$16 = 2 \cdot 8 + 0$$
$$gcd(96, 40) = 8$$

(b) 4 MARKS

$$8 = 40 - 2 \cdot 16 = (-2) \cdot 96 + 5 \cdot 40$$

hence $h = -2, k = 5$.

(c) 8 MARKS : 6 + 2

Yes, because $8|16$. A particular solution is $(-4, 10)$ and the set of all solutions is $(-4 + 5n, 10 - 12n)$ where $n \in \mathbb{Z}$.

The equation $96x + 40y = 5$ has no solutions because 8 does not divide 5.

2. 6 MARKS 3 + 3

Minimal polynomial : unique monic polynomial m_T satisfying:

1. $m_T(T) = 0$.
2. For any $f \in k[x]$ such that $f(T) = 0$, $\deg(g) \geq \deg(m_T)$.

T is diagonalisable if and only if m_T is a product of *distinct* polynomials of degree one.

3. 6 MARKS 2 + 2 + 2

- $k = \mathbf{R}$. $m_T(x) = (x-2)^2$. Not diagonalisable.
- $k = \mathbf{F}_2$. $m_T(x) = x^2$. Not diagonalisable.
- $k = \mathbf{F}_5$. $m_T(x) = x - 2$. Diagonalisable.

4. MARKS : 3 + 3 + 3

i Two 1×1 blocks.

ii One 2×2 block and one 1×1

iii One 3×3 block.

5. 10 MARKS

One finds canonical form:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Rank is 3, signature $(2,1)$.

Section B.

6. (a) 7 MARKS

Bézout's identity: there exists a pair of integers (h,k) such that

$$ah + bk = 1$$

Multiplying by c gives:

$$c = ach + bck$$

As $b|c$, $ab|ach$. As $a|c$, $ab|bck$ hence $ab|c$.

(b) (i), (ii) : 10 MARKS 2+5+3

(i). $f \in k[x]$ irreducible if whenever $f = gh$, g or h is a unit.

(ii).

(a)

TRUE

Suppose f has a root a in k, then $x - a$ divides f and as $\deg(f) > 1$, f is reducible. Conversely, suppose $f = gh$ where g and h are not units. As $\deg(f) = 2$, $\deg(g) = 1$ and g (and hence f) has a root in k.

(b)

FALSE

$x^4 + 1$ has no roots in $\mathbf{R}$ but

$$x^4 + 1 = (x^2 + \sqrt{2}x + 1)(x^2 - \sqrt{2}x + 1)$$

(iii) 8 MARKS 2 + 2 + 2 + 2

$$x^4 - 16 = (x^2 + 4)(x - 2)(x + 2)$$

In $\mathbf{C}[x]$:
$$x^4 - 16 = (x+2i)(x-2i)(x-2)(x+2)$$
Factors irreducible because degree one.
In $\mathbf{R}[x]$:
$$x^4 - 16 = (x^2+4)(x-2)(x+2)$$
x^2+4 irreducible because degree two and no roots in $\mathbf{R}$, other two have degree one hence irreducible.
In $\mathbf{F}_2[x]$:
$$x^4 - 16 = x^4$$
Factors are x - degree one hence irreducible.
In $\mathbf{F}_3[x]$:
$$x^4 - 1 = (x-1)(x+1)(x^2+1)$$
First two factors irreducible because degree one, the last one is degree 2" and has no roots in $\mathbf{F}_3$ hence irreducible.

7. (a) 7 MARKS : 3+4
The condition $\dim\ker(T_1 - 2\mathrm{Id}) = \dim\ker(T_2 - 2\mathrm{Id})$ implies that JNF for T_1 and T_2 has the same number of blocks.
(i). TRUE. The JNF for T_1 and T_2 has either two 1×1 or one 2×2 block.
(ii). FALSE. One can have two 2×2 blocks for T_1 and one 3×3 and one 1×1 for T_2.

(b) 12 MARKS : 3 + 3x3
$$T^2(A) = T(A+A^t) = (A+A^t)+(A+A^t) = 2T(A)$$
hence $T^2 = 2T$.
Let $f(x) = x^2 - x = x(x-2)$.
We proved that $f(T) = 0$ hence $m_T | f$.
For $k = \mathbf{R}, \mathbf{C}$, $m_T(x) = x(x-2)$ because both 0 and 2 are eigenvalues: $T(I) = 2I$ and for any non-zero anti-symmetric matrix A, $T(A) = 0$.
(alternatively, one can check that $T \neq 0$ and $T \neq 2\mathrm{Id}$)
T is diagonalisable when $k = \mathbf{R}, \mathbf{C}$.
For $k = \mathbf{F}_2$, $m_T(x) = x^2$. It suffices to check that $T \neq 0$. This is true because $T(E_{1,2}) = E_{1,2} + E_{2,1} \neq 0$ (notice that we used $n > 1$).
When $k = \mathbf{F}_2$, T is not diagonalisable.

6 MARKS
When $k = \mathbf{R}$, T is diagonalisable. The eigenvalues are 0 and 2.
The space $V_1(0)$ is the space of antisymmetric matrices, $\dim V_1(0) = \frac{n(n-1)}{2}$.
The space $V_1(2)$ is the space of symmetric matrices, $\dim V_1(2) = \frac{n(n+1)}{2}$.

It follows that

$$ch_T(x) = x^{\frac{n(n-1)}{2}}(x-2)^{\frac{n(n+1)}{2}}$$

8. (a)

(i). Bilinear, symmetric, positive definite

(ii). Not bilinear : $f(2,2) = 4$ and $f(1,2) + f(1,2) = 6$

(iii). Bilinear, not symmetric. $f(x,1) = 1$ but $f(1,x) = 0$.

(b) (i). 9 MARKS 4+5

(Existence) Let T^* be the linear map represented by $\bar{A}^t$. We'll prove that it is an adjoint of A.

$$\langle Tv, w\rangle = [v]^t A^t \overline{[w]} = [v]^t \overline{\bar{A}^t [w]}. = \langle v, T^* w\rangle.$$

Notice that here we have used that the basis is orthonormal : we said that the matrix of $<,>$ was the identity. (Uniqueness) Let T^*, T' be two adjoints. Then we have

$$\langle u, (T^* - T')v\rangle = 0.$$

for all $u, v \in V$. In particular, let $u = (T^* - T')v$, then $||(T^* - T')v|| = 0$ hence $T^*(v) = T(v)$ for all $v \in V$. Therefore $T^* = T'$.

(ii). 5 MARKS

Let $w \in W^\perp$ and $v \in W$. Then

$$\langle v, T^*(w)\rangle = \langle T(v), w\rangle = 0$$

because, by assumption, $T(v) \in W$ and $w \in W^\perp$.

Hence

$$T^*(W^\perp) \subset W^\perp.$$

(iv). 6 MARKS

Suppose $T^* = -T$. Let $\lambda \in \mathbf{C}$ be an eigenvalue. There exists a $v \neq 0$ such that $T(v) = \lambda v$.

$$< T(v), v >= \lambda < v, v >= - < v, \lambda v >= -\overline{\lambda} < v, v >$$

As $v \neq 0$, $< v, v > \neq 0$ hence $\overline{\lambda} = -\lambda$, it follows that λ is totally imaginary.

As an example, take T be the linear map represented by

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(c) MARKS

The matrix is symmetric, hence diagonalisable. Therefore, the minimal polynomial is $(x-5)(x+1)$.